**CALCULUS OF VARIATIONS AND OPTIMIZATION METHODS**

# Part I. Variations calculus

## Lecture 6. Lagrange Problem for functions of many variables

We have the method of the analysis for the minimization problem of integral functionals, which depends from unknown function and its derivative. This problem can be transformed to second order Euler differential equation. This method was being extended to the minimization problems with functionals, which depends from many unknown functions and from high derivatives of unknown functions. However, all unknown functions depend from one independent variable only. However, there exist many practical minimization problems with unknown functions of many variables. We will try to extend the previous results to these problems. We will consider the problem of the minimization of Dirichlet integral and the problem of the oscillation of a string as examples.

### 6.1. Problem statement

Let Ω be the set on the plane with boundary *S*. We consider the functional



where *F* is a given function, and unknown function  satisfies the boundary conditions

  (6.1)

with given function *ϕ*.

**Problem 6**.1. *Find the function* , *which minimize the functional I with boundary conditions* (6.1).

### 6.2. Ostrogradsky equation

We will try to use the known techniques for solving the given problem.

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| **Question**: *How we can solve this problem?* |

Let *u* be a solution of our problem. Determine the function of one variable



where *σ* is a number, and *h* is a smooth enough function on the interval .

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| **Question**: *What are the boundary conditions for the function h?* |

The function *h* satisfies the homogeneous boundary conditions

 . (6.2)

Then the function  is admissible, because it satisfies boundary condition (6.1).

The functional *I* has the minimum at the point *u* if and only if the number 0 is the point of the minimum for the function *f*. So we try to calculate the derivative of the function *f* at zero and equal it to 0. Suppose the function *F* is smooth enough. Denote by  the partial derivatives of the function *F* with respect to the third, fourth and fifth arguments.

**Lemma 6.1**. *The derivative of the function f at the zero is equal to the integral*



**Proof**. We find the value



Using Taylor formula we get



where  as  Then we obtain



After devising by *σ* and passing to the limit as  we get



We have the equalities





So we get

  (6.3)

It is known *Green formula*



which is true for all smooth enough functions  and . We chose here



So we have



The term in the right side of this equality is equal to zero because of the boundary condition (6.2). So the equality (6.3) can be transformed to



It completes the proof of the lemma.

The term in right side of this equality is *the variation of the functional*, it is denoted by  So the equality

  (6.4)

is true for all functions *h*, which satisfies the boundary condition (6.2).

It is known the multidimensional analogue of the *Fundamental Lemma of Calculus of variations*.

**Lemma 6.2**. *If the equality*



*with continuous function  is true for all continuous function  then the function g is equal to zero.*

Using Lemma 6.2 and equality (6.4) we get the following result.

**Theorem 6.1**. *If the smooth enough function u is a solution of Problem* 6.1, *then it satisfies* ***Ostrogradsky equation***

  (6.5)

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| **Question**: *What kind of equations is the equality* (6.5)? |

Ostrogradsky equation is the second order partial differential equation. It is considered with boundary condition

 . (6.6)

Hence we have boundary problem (6.5), (6.6) as a necessary condition of extremum for Problem 6.1. We will consider applications of these results.

### 6.3. Dirichlet integral

Consider an example. The integral



is called the *Dirichlet integral.* The function *f* is given here. We consider the problem of its minimization on set of functions  which satisfy the boundary condition (6.1).

The function *F* is equal here to



Find its partial derivatives



Then Ostrogradsky’s equation (6.5) is transformed to

  (6.7)

The equation (6.7) is called the *Poisson equation*, and the boundary problem (6.6), (6.1) is called the *Dirichlet problem*. Therefore, the solution of the minimization problem for Dirichlet integral with boundary condition (6.1) satisfies Dirichlet problem (6.6), (6.1).

### 6.4. The oscillation of the string

Consider the oscillation of a string. It is described by the function  it is the state of the string in the point *x* at the time *t*. This is the deviation of the spring from the equilibrium state. We will analyze the movement of the string with using of the law of least action. The total energy of the string movement is minimal.

Determine the action *S* for the string oscillation. It is the sum of the kinetic energy *K* and the work *W*. The kinetic energy is determined by the velocity. It is equal to

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where *m* is the mass, and  is the time derivative of the function *u*; that is the velocity of the movement.

The string is characterized by the density *ρ*. Let the string by a one-dimensional object. The density is the mass of the unit length of the string in this case. Then the mass of the string with length Δ*x* is *ρ*Δ*x*. So the kinetic energy is equal to

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This formula is true if the characteristics of the string do not change with respect to its length. If it is not constants, then this formula can be true for small enough length *dx* only. So we can calculate the value

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Then the kinetic energy of the string with length *L* is equal to the value

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Determine now the work. This is the product of the force and the way, i.e.

*W = FS.*

 Let the body has the velocity *ut* . Its way during the time Δ*t* is equal to *ut*Δ*t*. If we have the force *F*, then the work is equal to

*W = F* *ut*Δ*t*.

Determine the value of the force *F*. We suppose that there exists a unique force of the tension here. The tension *k* is directed at the tangent of the string (see Figure 6.1). We consider the small oscillation, so the tension is a constant; it is the parameter of the system. We analyze the oscillation in the perpendicular direction to the axe *x* only. Therefore, we use the projection of the tension to the axe *u*. We obtain the equality

*F* = *k* sin *α*,

where *α* is the angle of the tangent line of the curve (see Figure 6.1).

We consider the small oscillations only. In this case the sine is equal to the tangent approximately. However the tangent for the function *u* of the argument *x* is the derivative  Hence, we find

*F*  = *k* tg*α* = *k* *ux*.

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Figure 6.1. The tension *k* is directed at the tangent of the curve.

Determine the force *F* of the tension of the string with the begin  and the end . The force *F* is the difference between the forces *F*1 and *F*2 on the begin and the end of the string (see Figure 6.2). So we find



Then we find the potential energy





Figure 6.2. The force of the tension for the string with the length *Х*.

The values of the derivatives  and  are constants here. Therefore, the last formula can be true if we have small enough intervals *dx* and *dt* only. Now we obtain the equality

*dW* = (*k* *ux*)*x* *ut**dx* *dt*.

Then the action of the movement of the string with the length *L* from the time 0 to the time *t* is equal to



We have the equality



Suppose that the ends of the string are fixed. So its velocity is equal to zero. Therefore, we get

*ut*(0,*t*) = 0, *ut*(*X*,*t*) = 0.

Than we find the potential energy



We know the formula



So we find



Suppose that the string is in the equilibrium state at the time *t* = 0, so  Hence we find



Then the action of the oscillation in the concrete time is equal to



So the energy of oscillation from the time 0 to the time *T* is equal to



By the law of least action, the value of the energy need be minimal. Therefore, we have the problem of the minimization of the functional *I.* Hence, the law of movement can be determined as Ostrogradsky equation for this problem. Hence, we have the equation



It can be transform to

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If the string is homogeneous, then *k* is constant. Denote  It is the parameter of the process. Therefore, we obtain the equation



It called ***the equation of the string******oscillation***. It is one of the important equations of the mathematical physics.

### Outcome

* The solution of Lagrange problem with unknown functions of many variables satisfies Ostrogradsky equation with boundary conditions.
* Ostrogradsky equation is second order partial differential equation.
* The solution of the obtained boundary problem can be the solution of Lagrange problem, but may be it is not its solution.
* The problem of the minimization of Dirichlet integral is an application of this theory; it can by transformed to Dirichlet problem for Poisson equation.
* The problem of bending of the elastic beam is an example of this theory.

### Task. Ostrogradsky equation for the three dimensional case

We consider the functional



where Ω is three-dimensional set, *F* is given function. Boundary conditions are given too. Write the corresponding Ostrogradsky equation



The values of the function *F*:

1. 
2. 
3. 
4. 
5. 
6. 
7. 
8. 
9. 
10. 

### Task 2. Ostrogradsky equation for the three dimensional case

We consider the functional



where Ω is three-dimensional set, *F* is given function,  Boundary conditions are given too. Write the corresponding Ostrogradsky equations



The values of the function *F*:

1. 
2. 
3. 
4. 
5. 
6. 
7. 
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12. 
13. 
14. 
15. 

**Literature**

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### Next step

We have the standard method for solving the problems of minimization for integral functionals with one or many unknown functions. These functionals can depend from the unknown functions and its derivatives. We considered the functions of one or many variables. We analyzed the minimization problems with fixed boundary conditions only. We will try to extend our results to the problem without boundary conditions.